ON SUMS OF LARGE SETS OF INTEGERS(1)

BY
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1. Let n be a fixed positive integer. Consider $k \geqq 3$ sets $A_1, \cdots, A_k$ of non-negative integers, each containing zero, such that

$$n \notin C, \tag{1.1}$$

where

$$C = A_1 + \cdots + A_k$$

consists of all the integers which can be written as a sum $a_1 + a_2 + \cdots + a_k$ with $a_1 \in A_1, \cdots, a_k \in A_k$. If S is a set of integers, we write

$$S(x, y) = \sum_{x < s \leqq y;\, s \in S} 1.$$

It is well known that (1.1) implies

$$\sum_{i=1}^{k} A_i(0, n) \leqq k \frac{n-1}{2}. \tag{1.2}$$

Moreover, (see §3), if the equality sign holds in (1.2) then necessarily

$$C(0, n) = \frac{n-1}{2},$$

(hence, n must be odd). Thus, (1.2) can be improved if in addition to (1.1) we require that

$$\{0, 1, 2, \cdots, n-1\} \subset C, \tag{1.3}$$

that is, n is precisely the smallest positive integer not in C. Let

$$\phi_k(n) = \min \left(k \frac{n-1}{2} - \sum_{i=1}^{k} A_i(0, n) \right), \tag{1.4}$$

where $(A_1, \cdots, A_k)$ ranges over the k-tuples satisfying (1.1) and (1.3). It was shown by Erdös and Scherk [1] that (for $k \geqq 3$)

$$\alpha_k n^{(k-1)/k} \leqq \phi_k(n) \leqq \beta_k n^{(k-1)/k}, \tag{1.5}$$

where

$$\alpha_k = \{2^{k/2+4}(k-1)!\}^{-1}, \qquad \beta_k = (k+1)2^{2k-3}.$$

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This result will be sharpened to:

THEOREM 1.1. *One has, for each $k \geqq 3$, that*

$$\lim_{n\to\infty} \phi_k(n)n^{-(k-1)/k} = k2^{-(k-1)/k}. \tag{1.6}$$

Note that $\phi_2(n) = 0$; for, take $A_1 = \{0,1,\cdots,m\}$ and $A_2 = \{0,1,\cdots,n-1-m\}$; further on, we shall always assume that $k \geqq 3$. As will be seen, the result (1.6) can be generalized to other conditions of the type (1.3). One of the important tools is Lemma 3.2. Its proof is given in §5 and is based upon an isoperimetric inequality which seems to be of independent interest. In §3, using Lemma 3.2, we easily obtain an inequality of the type (1.5) with $\beta_k = 2\alpha_k$. The proof of the stronger result (1.6) is given in §4.

2. Let us first establish an upper bound on $\phi_k(n)$, namely,

$$\phi_k(n) < k\left(1 + \left(\frac{n}{2}\right)^{1/k}\right)^{k-1}. \tag{2.1}$$

By $\phi_k(1) = 0$, we may assume $n \geqq 2$. Put

$$p = 1 + \left[\left[\frac{n}{2}\right]^{1/k}\right] \qquad (p \geqq 2);$$

thus, p is the smallest positive integer satisfying

$$\left[\frac{n}{2}\right] \leqq p^k - 1.$$

Letting

$$D_i = \{0, p^{i-1}, 2p^{i-1}, \cdots, (p-1)p^{i-1}\} \qquad (i = 1,\cdots,k),$$

each integer $0 \leqq y \leqq p^k - 1$ has a unique representation as a sum $y = d_1 + \cdots + d_k$ with $d_i \in D_i$. In particular,

$$\left\{0,1,\cdots,\left[\frac{n}{2}\right]\right\} \subset D_1 + D_2 + \cdots + D_k. \tag{2.2}$$

If n is small it may happen that $n \in D_1 + \cdots + D_k$, that is,

$$n = \sum_{i=1}^{j} \varepsilon_i p^{i-1},$$

where $1 \leqq j \leqq k$, $\varepsilon_i p^{i-1} \in D_i$, $\varepsilon_j \geqq 1$. But then

$$n \leqq \varepsilon_j p^{j-1} + (p-1)\sum_{i=1}^{j-1} p^{i-1} < 2\varepsilon_j p^{j-1}.$$

Now, modify D_j by canceling the single element $\varepsilon_j p^{j-1} > n/2$ (the other D_i remain unchanged). This does not effect property (2.2). Moreover, we now have that

$$n \notin D_1 + \cdots + D_k.$$

Finally, we form A_i $(i = 1, \cdots, k)$ as the union of D_i and the collection of all integers $n/2 < x < n$ such that $n - x \notin E_i$, where

$$E_i = D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_k.$$

Clearly, the resulting k-tuple satisfies (1.1). It also satisfies (1.3). By (2.2), it suffices to show that $x \in A_i$ for some i, where x is a given integer satisfying $n/2 < x < n$. If not, then the *positive* integer

$$n - x = \sum_{i=1}^{k} \eta_i p^{i-1} \qquad \text{(say)},$$

$(\eta_i p^{i-1} \in D_i,\ i = 1, \cdots, k)$, would belong to each set E_i, that is, $\eta_i = 0$ for all i, which is a contradiction.

Finally, from the above construction,

$$A_i(0,n) \geqq A_i\left(\frac{n}{2}, n-1\right) \geqq \left[\frac{n-1}{2}\right] - E_i\left(0, \frac{n-1}{2}\right).$$

But each D_h has at most p elements, thus, E_i has at most p^{k-1} elements; further, $0 \in E_i$, hence,

$$k\frac{n-1}{2} - \sum_{i=1}^{k} A_i(0,n) \leqq k\frac{n-1}{2} - k\left[\frac{n+1}{2}\right] + kp^{k-1} < kp^{k-1}.$$

In view of (1.4) and the definition of p, this proves (2.1).

Note that, for n large, the above sets A_i contain only very few elements smaller than $n/2$. As will be seen, this is always the case for k-tuples satisfying (1.1), (1.3), which nearly attain the minimum in (1.4).

3. It remains to derive lower bounds on $\phi_k(n)$. Let $A_1, \cdots, A_k$ denote given sets of non-negative integers, each containing zero. Put

$$C = A_1 + \cdots + A_k \tag{3.1}$$

and

$$B_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_k \qquad (i = 1, \cdots, k). \tag{3.2}$$

By $0 \in A_h$ $(h = 1, \cdots, k)$, we have

$$0 \in A_j \subset B_i \subset C \qquad \text{if } j \neq i. \tag{3.3}$$

Thus, C can be written as the *disjoint* union

$$C = D \cup \Delta \cup C^*, \tag{3.4}$$

where

(3.5) $$D = A_1 \cap A_2 \cap \cdots \cap A_k,$$

(3.6) $$\Delta = (B_1 \cup B_2 \cup \cdots \cup B_k) \cap \bar{D},$$

and

(3.7) $$C^* = C \cap \bar{B}_1 \cap \bar{B}_2 \cap \cdots \cap \bar{B}_k;$$

here the bar denotes complementation.

LEMMA 3.1. *One has*

(3.8) $$\Delta = \bigcup_{i=1}^{k} (B_i \cap \bar{A}_{j_i})$$

for each permutation $(j_1, \cdots, j_k)$ *of* $(1, \cdots, k)$.

Proof. Let $f_j(x)$ denote the characteristic function of A_j, that is, $f_j(x) = 1$ if $x \in A_j$, $f_j(x) = 0$ otherwise. Similarly, let $g_i(x)$ denote the characteristic function of $\bar{B}_i$. By (3.3),

$$g_i(x) f_j(x) = 0 \qquad \text{if } i \neq j;$$

thus,

$$g_i(x) g_j(x) f_h(x) = 0, \qquad g_h(x) f_i(x) f_j(x) = 0,$$

whenever i $\neq j$. Consequently, by $k \geqq 3$,

$$\prod_{i=1}^{k} (g_i(x) + f_{j_i}(x)) = \prod_{i=1}^{k} g_i(x) + \prod_{j=1}^{k} f_j(x).$$

Given x, the latter right-hand side is equal to 0 if and only if $x \in \Delta$, by (3.5) and (3.6). The left-hand side is equal to 0 if and only if $x \in B_i \cap \bar{A}_{j_i}$ for some $i = 1, \cdots, k$. This proves (3.8).

A perhaps more intuitive equivalent proof of Lemma 3.1 was suggested by the referee. Clearly, Δ is precisely the union of the k^2 sets $B_h \cap \bar{A}_{j_i}$ and thus contains the right-hand side Δ^1 (say) of (3.8). Conversely, let $x \in B_h \cap \bar{A}_{j_i}$. In proving $x \in \Delta^1$, we may assume that

$$x \notin B_i, \qquad x \in A_{j_h}.$$

It follows by (3.3) that $x \in B_r$ when $r \neq j_h$, thus, $i = j_h$, and further that $x \notin A_{j_r}$ when $j_r \neq i = j_h$, that is, when $r \neq h$. Therefore, $x \in B_r \cap \bar{A}_{j_r}$ for each index r with $r \neq i$ and $r \neq h$, in particular, $x \in \Delta^1$.

The following result is an immediate consequence of Theorem 5.1, which will be proved in §5. Here, and in the sequel, if E is a set then $[E]$ denotes the number of elements in E.

LEMMA 3.2. *Let E be an arbitrary subset of C and let*

(3.9) $$F_i = \{b_i \in B_i : a_i + b_i \in E \text{ for some } a_i \in A_i\}.$$

Then

$$[E] \leqq ([F_1][F_2] \cdots [F_k])^{1/(k-1)}.$$

This in turn implies

(3.10) $$[E]^{(k-1)/k} \leqq \frac{1}{k} \sum_{i=1}^{k} [F_i].$$

Let x denote a fixed positive integer, and let us apply Lemma 3.2 with

$$E = \{c \in C^* : c \leqq x\}.$$

By (3.3) and (3.7), $0 \notin E$, hence

$$[E] = C^*(0, x).$$

Consider a fixed element $c \in E \subset C^* \subset C$. By (3.2) and (3.7), if

$$c = a_1 + a_2 + \cdots + a_k, \qquad a_i \in A_i,$$

then necessarily $a_i > 0$ $(i = 1, \cdots, k)$. Letting $c = a_i + b_i$, one has $b_i \in F_i \subset B_i$ and $0 < b_i \leqq c - 1 \leqq x - 1$. Moreover, in each representation

$$b_i = a_1' + \cdots + a_{i-1}' + a_{i+1}' + \cdots + a_k',$$

(with $a_j' \in A_j$ for $j \neq i$), one necessarily has $a_j' > 0$ for each $j \neq i$. In particular, by $k \geqq 3$, $b_i \notin A_j$ for each $j \neq i$. Hence, by (3.9),

$$F_i \subset \{b_i \in B_i : 0 < b_i \leqq x - 1,\ b_i \notin A_j \text{ for each } j \neq i\}.$$

Thus, (3.10) yields

LEMMA 3.3. *For each positive integer x,*

(3.11) $$C^*(0,x)^{(k-1)/k} \leqq \frac{1}{k} \sum_{i=1}^{k} \left\{ B_i \bigcap_{j \neq i} \bar{A}_j \right\}(0, x-1).$$

Let us now assume that

(3.12) $$n \notin C.$$

This is equivalent to

(3.13) $$n \notin A_i + B_i \qquad (i = 1, \cdots, k).$$

Hence, by (3.3), $n \notin A_i$ and $n \notin B_i$, thus,

$$A_i(0,n) = A_i(0, n-1), \qquad B_i(0,n) = B_i(0, n-1).$$

By (3.13), if $x < y$ are real numbers and $b \in B_i$, $x < b \leqq y$ then $n - b \notin A_i$, $n - [y] \leqq n - b < n - [x]$. Therefore, (3.14) $$B_i(x,y) \leqq [y] - [x] - A_i(n-1-[y],\, n-1-[x]).$$

In particular,

$$B_i(0,n-1) \leqq n-1-A_i(0,n-1).$$

Moreover, by (3.3), $A_{i+1} \subset B_i$ $(A_{k+1} = A_1)$; consequently,

(3.15) $$\sum_{i=1}^{k} \{B_i \cap \bar{A}_{i+1}\}(0,n) \leqq 2\rho$$

(thus, $\rho \geqq 0$), where ρ is defined by

(3.16) $$\sum_{i=1}^{k} A_i(0,n) = k\,\frac{n-1}{2} - \rho.$$

By (3.8) and (3.15),

(3.17) $$\Delta(0,n) \leqq 2\rho.$$

By (3.11) and (3.15),

$$C^*(0,n) \leqq (2\rho/k)^{k/(k-1)}.$$

Finally, by (3.5) and (3.16),

$$D(0,n) \leqq \min_i A_i(0,n) \leqq \frac{n-1}{2} - \frac{\rho}{k}.$$

Thus, by (3.4):

THEOREM 3.4. *Suppose that* (3.12) *and* (3.16) *hold. Then*

(3.18) $$C(0,n) \leqq \frac{n-1}{2} + \left(2 - \frac{1}{k}\right)\rho + (2\rho/k)^{k/(k-1)}.$$

If $\rho = 0$ then $C(0,n) \leqq (n-1)/2$. But $A_i \subset C$; thus, by (3.16), $\rho = 0$ implies that below n the sets A_i and C constitute one and the same set A, say, consisting of 0 and further $(n-1)/2$ positive integers $< n$. Moreover, $a, a' \in A$ implies that either $a + a' \in A$ or $a + a' > n$. Conversely, if n is odd and A is any such set then the choice $A_i = A$ $(i = 1, \cdots, k)$ yields a k-tuple satisfying $n \notin C$ and $\rho = 0$. Typical examples are

$$A = \{0, m+1, m+2, \cdots, 2m\}$$

and

$$A = \{0, 2, 4, \cdots, 2m\},$$

where $m = (n-1)/2$. In general, $n = 2m+1 \notin A$ implies that either $m \notin A$ or $m + 1 \notin A$ which in turn implies

(3.19) $$A\left(0, \frac{n-1}{2}\right) \leqq \frac{n-1}{4} \qquad \text{(when } \rho = 0).$$

If (besides (3.12)) we assume that

$$\{0,1,2,\cdots,n-1\} \subset C$$

then $C(0,n) = n-1$. In the more general case where we assume that

$$C(0,n) \geqq \left(\frac{1}{2}+\frac{1}{p}\right)(n-1) \qquad (p \geqq 2), \tag{3.20}$$

(3.18) yields

$$\left(\frac{2\rho}{k}\right)^{k/(k-1)} \geqq \frac{n-1}{p} - \frac{2k-1}{2}\left(\frac{n-1}{p}\right)^{(k-1)/k}. \tag{3.21}$$

After all, if (3.21) were false then

$$\left(2-\frac{1}{k}\right)\rho = \frac{2k-1}{2}\,\frac{2\rho}{k} < \frac{2k-1}{2}\left(\frac{n-1}{p}\right)^{(k-1)/k}.$$

It follows from (3.21) (with $p=2$), (3.16) and (1.4) that

$$\phi_k(n) \geqq \frac{1}{2}k\left(\frac{n-1}{2}\right)^{(k-1)/k} - O(n^{(k-2)/k}). \tag{3.22}$$

Together with (2.1) this yields for n large an estimate of the type (1.5) with

$$\beta_k = k\left(\frac{1}{2}\right)^{(k-1)/k}, \qquad \alpha_k = \frac{1}{2}\beta_k.$$

4. In this section, the lower bound (3.22) will be improved so as to demonstrate Theorem 1.1. As a byproduct, we obtain some additional information on the structure of the "optimal" k-tuples.

Let again $A_1,\cdots,A_k$ be sets of non-negative integers such that $0 \in A_i$ and

$$n \notin C = A_1 + \cdots + A_k. \tag{4.1}$$

Further, $\rho \geqq 0$ will always be defined by

$$\sum_{i=1}^{n} A_i(0,n) = k\,\frac{n-1}{2} - \rho. \tag{4.2}$$

Applying (3.14) with $x=0$, $y=(n-1)/2$, we have

$$B_i\left(0,\frac{n-1}{2}\right) \leqq \left[\frac{n-1}{2}\right] - A_i\left(\left[\frac{n}{2}\right], n-1\right). \tag{4.3}$$

By $A_{i+1} \subset B_i, (A_{k+1} = A_1)$, this yields

$$\sum_{i=1}^{k}\{B_i \cap \bar{A}_{i+1}\}\left(0,\frac{n-1}{2}\right) \leqq k\left[\frac{n-1}{2}\right] - \sum_{i=1}^{k}(A_i(0,n-1)-\varepsilon_i),$$

where $\varepsilon_i = 1$ if $n/2 \in A_i$, $\varepsilon_i = 0$ otherwise. But $\varepsilon_i = 1$ can only happen when n is even and then only for at most one index i, by $n \notin C$. Therefore, using (4.2),

$$\sum_{i=1}^{k} \{B_i \cap \bar{A}_{i+1}\}\left(0, \frac{n-1}{2}\right) \leqq \rho. \tag{4.4}$$

It follows by (3.11) that

$$C^*\left(0, \frac{n+1}{2}\right) \leqq (\rho/k)^{k/(k-1)} \tag{4.5}$$

and by (3.8) that

$$\Delta\left(0, \frac{n-1}{2}\right) \leqq \rho. \tag{4.6}$$

Remember (cf. (3.17)) that

$$\Delta(0, n-1) \leqq 2\rho. \tag{4.7}$$

Definition. With q as a positive integer, let S_q denote the set of all integers $1 \leqq m \leqq (n-1)/2$ admitting at least one representation of the form

$$m = a_1 + a_2 + \cdots + a_q \text{ with } a_j \in \bigcup_{i=1}^{k} A_i \ (j = 1, \cdots, q). \tag{4.8}$$

Let further $\lambda_q \geqq 0$ denote the number of integers $1 \leqq m \leqq (n-1)/2$ with $m \notin S_q$. Note that, by $0 \in A_i$, we have $S_{q+1} \supset S_q$ and $\lambda_{q+1} \leqq \lambda_q$.

Theorem 4.1. *We have for each positive integer q that*

$$D\left(0, \frac{n-1}{2}\right) \leqq \left(1 + \frac{1}{\sqrt{2}}\right)\lambda_q + (4 + 2\sqrt{2})q\rho. \tag{4.9}$$

Let us first establish a few lemmas.

Lemma 4.2. *Let $m \in S_q$. Then*

(i) *There are at most $2q\rho$ elements $d \in D$ such that $0 \leqq d \leqq (n-1)/2$ and $d + m \notin D$.*

(ii) *There are at most $q\rho$ elements $d \in D$ such that $0 \leqq d \leqq (n-1)/2 - m$ and $d + m \notin D$.*

Proof. Let (4.8) be a fixed representation of $m \in S_q$. Consider an element $d \in D$ such that $0 \leqq d \leqq (n-1)/2$ and $d + m \notin D$. There exists a smallest index $1 \leqq r \leqq q$ such that

$$d' = d + a_1 + \cdots + a_{r-1} \in D \text{ and } d' + a_r \notin D. \tag{4.9}$$

If $a_r \in A_j$ and $h \neq j$ then $d' + a_r \in B_h$ (choose $1 \leqq i \leqq k$ distinct from j and h and observe that $d' \in D \subset A_i$). Hence, by (3.6), $d' + a_r \in \Delta$. Moreover, $d' + a_r \leqq d + m \leqq n - 1$; thus, by (4.7), we have for each fixed $1 \leqq r \leqq q$ that

(4.9) can happen for at most 2ρ elements d. This proves assertion (i). In a similar way, (4.6) yields assertion (ii).

LEMMA 4.3. *Let* $m \in S_q$. *Then*

$$D\left(\frac{n-1}{2} - m, \frac{n-1}{2}\right) \leqq \frac{m}{2} + q\rho. \tag{4.10}$$

Hence,

$$D\left(0, \frac{n-1}{2} - m\right) \geqq N - \frac{m}{2} - q\rho, \tag{4.11}$$

where

$$N = D\left(0, \frac{n-1}{2}\right). \tag{4.12}$$

Proof. Consider the sets

$$D' = \left\{d \in D : \left[\frac{n+1}{2}\right] - m \leqq d \leqq \left[\frac{n}{2}\right]\right\}$$

and

$$D'' = \left\{d \in D : \left[\frac{n+1}{2}\right] \leqq d \leqq \left[\frac{n}{2}\right] + m\right\}.$$

Notice that

$$\left[\frac{n+1}{2}\right] + \left[\frac{n}{2}\right] = n.$$

Hence, by $n \notin C$, $D + D \subset C$, one has

$$[D'] + [D''] \leqq m$$

(if n is even, then $n/2 \notin D$).

On the other hand, by Lemma 4.2, one has for all but $2q\rho$ elements $d \in D'$ that $d + m \in D''$, thus, $[D''] \geqq [D'] - 2q\rho$; therefore,

$$m \geqq [D'] + [D'] - 2q\rho,$$

implying (4.10).

LEMMA 4.4. *If* $m \in S_q$ *then there are at least*

$$N + 1 - \frac{m}{2} - 2q\rho$$

pairs of elements d, d' *in D such that*

$$0 \leqq d < d' \leqq \frac{n-1}{2}, \quad d' - d = m. \tag{4.13}$$

Proof. By (4.11) and $0 \in D$, there are at least $N+1-m/2-q\rho$ elements $d \in D$ with $0 \leqq d \leqq (n-1)/2-m$. By Lemma 4.2, we have for all but at most $q\rho$ of these elements that $d' = d + m \in D$.

The following generalizes (3.19).

LEMMA 4.5. *One has*

$$N \leqq \frac{n-1}{4} + 2\rho. \tag{4.14}$$

Proof. Let d_0 denote the largest element $d_0 \in D$ with $0 < d_0 \leqq (n-1)/2$ (if no such elements exist then $N = 0$). Notice that (4.13) with $m = d_0$, $d' \in D$ is only possible with $d = 0$, $d' = d_0$. Hence, applying Lemma 4.4 with $m = d_0 \in S_1$, one obtains that

$$N + 1 - \frac{d_0}{2} - 2\rho \leqq 1,$$

proving (4.14).

Proof of Theorem 4.1. There are precisely $N+1$ elements $d \in D$ with $0 \leqq d \leqq (n-1)/2$, hence, precisely $\frac{1}{2}N(N+1)$ pairs d, d' in D with $0 \leqq d < d' \leqq (n-1)/2$. It follows by Lemma 4.4 that

$$\frac{1}{2}N(N+1) \geqq \sum_{m \in T}\left(N+1-\frac{m}{2}-2q\rho\right) = [T](N+1-2q\rho) - \sum_{m \in T}\frac{m}{2} \tag{4.15}$$

is true for any subset T of S_q. We shall choose

$$T = \{m \in S_q : 1 \leqq m \leqq r\},$$

where r denotes the integer defined by

$$r = 2N - 4q\rho. \tag{4.16}$$

One may assume that $r > 0$; otherwise, (4.9) is obvious. Further, $r \leqq (n-1)/2$ by (4.14) and $q \geqq 1$. It follows from the definition of λ_q that

$$[T] = r - \lambda'_q \text{ with } 0 \leqq \lambda'_q \leqq \lambda_q.$$

Moreover, by (4.15),

$$\frac{1}{2}N(N+1) \geqq (r-\lambda'_q)(N+1-2q\rho) - \sum_{m=1}^{r}\frac{m}{2} + \sum_{m=1}^{\lambda'_q}\frac{m}{2}.$$

Substituting (4.16), one obtains the inequality

$$\frac{1}{2}N^2 - \alpha N + \beta \leqq 0, \tag{4.17}$$

where

$$\alpha = \gamma - 1, \quad \beta = \frac{1}{4}\gamma(\gamma - 3).$$

Here,

$$\gamma = 4q\rho + \lambda_q', \quad \text{thus,} \quad 0 \leqq \gamma \leqq 4q\rho + \lambda_q.$$

Note that

$$0 \leqq \alpha^2 - 2\beta = \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma + 1 \leqq \left(\frac{1}{\sqrt{2}}\gamma + 1\right)^2.$$

Consequently, by (4.17),

$$N \leqq \alpha + \sqrt{(\alpha^2 - 2\beta)} \leqq (\gamma - 1) + \left(\frac{1}{\sqrt{2}}\gamma + 1\right) \leqq \left(1 + \frac{1}{\sqrt{2}}\right)(4q\rho + \lambda_q).$$

This proves Theorem 4.1.

Theorem 1.1 is an easy consequence of Theorem 4.1. As before, suppose $n \notin C$ and let ρ be defined by (4.2). Instead of (1.3), let us merely assume that

$$\left\{0, 1, \cdots, \left[\frac{n-1}{2}\right]\right\} \subset C,$$

or still less, that

$$C\left(0, \frac{n-1}{2}\right) \geqq \left[\frac{n-1}{2}\right] - \lambda(n), \tag{4.18}$$

where $\{\lambda(n)\}$ is a *given* sequence of positive numbers such that

$$\lim_{n \to \infty} \frac{\lambda(n)}{n} = 0. \tag{4.19}$$

Noting that $C \subset S_k$, we have that (4.18) implies $\lambda_k \leqq \lambda(n)$. Hence, by (3.4), (4.5), (4.6) and (4.9) (with $q = k$),

$$\left[\frac{n-1}{2}\right] - \lambda(n) \leqq \left(1 + \frac{1}{\sqrt{2}}\right)\lambda(n) + (4 + 2\sqrt{2})k\rho + \rho + (\rho/k)^{k/(k-1)}.$$

It follows that

$$(\rho/k)^{k/(k-1)} \geqq \frac{n}{2} - o(n). \tag{4.20}$$

Thus,

$$\liminf_{n \to \infty} n^{-(k-1)/k} \min \rho \geqq k\left(\frac{1}{2}\right)^{(k-1)/k}$$

(here, for each fixed n, the minimum is subject to $n \notin C$ and (4.18)). In view of (1.4), (2.1) and (4.2), this proves Theorem 1.1.

Notice that a k-tuple $\{A_1, \cdots, A_k\}$ which satisfies $n \notin C$ and (4.18), and which is nearly optimal in the sense that

$$\rho = o(n),$$

satisfies $D(0,(n-1)/2) = o(n)$, by (4.9) with $q = k$; hence, $A_i(0,(n-1)/2) = o(n)$, by (3.3), (3.6) and (4.6); hence,

$$A_i\left(\frac{n-1}{2}, n-1\right) = \frac{n-1}{2} - o(n) \qquad (i = 1,\cdots,k);$$

therefore, $C(0,n-1) = n-1-o(n)$. In other words, for optimal k-tuples, the assumption (4.18) *nearly* implies the much stronger (1.3).

If even $(\rho/k)^{k/(k-1)} \sim n/2$ then (4.5) holds nearly with the equality sign, suggesting that the part of C^* below $n/2$ does not deviate much from a direct sum with nearly disjoint components of about equal size, compare Theorem 5.1 and the way it was used in the proof of (4.5).

The reasoning leading to (4.20) can easily be adapted in proving the following result.

Let $\{\lambda(n)\}$ be as in (4.19) and let $\{q(n)\}$ be a sequence of positive integers satisfying $q(n) \to \infty$, $q(n) = o(n^{1/k})$. Let further $0 < \varepsilon \leqq 1$ be a given positive constant and let

$$\psi_k(n,\varepsilon) = \min\left\{k\,\frac{n-1}{2} - \sum_{i=1}^{n} A_i(0,n)\right\}, \tag{4.21}$$

where $\{A_1,\cdots,A_k\}$ ranges over the k-tuples of sets A_i of non-negative integers such that $0 \in A_i$, $n \notin C$,

$$C\left(0,\frac{n-1}{2}\right) \geqq \varepsilon\left[\frac{n-1}{2}\right] \tag{4.22}$$

and

$$\lambda_{q(n)} \leqq \lambda(n). \tag{4.23}$$

THEOREM 4.6. *With the above notations,*

$$\lim_{n\to\infty} \psi_k(n,\varepsilon)n^{-(k-1)/k} = k(\varepsilon/2)^{(k-1)/k}. \tag{4.24}$$

Here, an upper bound on ρ is easily obtained from the construction in §2, by replacing there p by the smallest integer satisfying

$$\varepsilon\frac{n}{2} \leqq p^k - 1$$

(the resulting k-tuple has $\lambda_{kh} = 0$ as soon as $h[\varepsilon n/2] \geqq (n-1)/2$).

The restriction (4.23) cannot be omitted. For, take each A_i as the set A obtained from $\{[(n+1)/2],\cdots,n-1\}$ by adding a block of multiples rt of t immediately below $[(n+1)/2]$ and deleting the corresponding elements $n-rt$; here, t is a positive integer not dividing n. In this example, λ_q is independent of q.

5. Let G be an arbitrary commutative semigroup, written additively (that is, a set together with an associative and commutative addition). Lemma 3.2 is an immediate consequence of:

THEOREM 5.1. *Be given $k \geqq 2$ finite and nonempty subsets $A_1, \cdots, A_k$ of G. Put*

$$C = A_1 + \cdots + A_k, \quad B_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_k$$

$(i = 1, \cdots, k)$. Let E be any subset of C and put

$$B_i^* = \{b_i \in B_i : a_i + b_i \in E \text{ for some } a_i \in A_i\}. \tag{5.1}$$

Then

$$[E] \leqq ([B_1^*][B_2^*] \cdots [B_k^*])^{1/(k-1)}. \tag{5.2}$$

Here, the equality sign holds if and only if E is the direct sum of the k sets

$$A_i^* = \{a_i \in A_i : a_i + b_i \in E \text{ for some } b_i \in B_i\}.$$

More precisely, E consists of all elements c of the form $c = a_1^* + \cdots + a_k^*$ $(a_i^* \in A_i^*)$; moreover, given $c \in E$, this representation is unique. In particular, $[E] = [A_1^*] \cdots [A_k^*]$; if a point a_0 is in two A_i^* it is in all A_i^*, and there can be only one such a_0.

For the proof of Theorem 5.1 we shall need the following lemma, which is contained in the more general Theorem 5.3. In the sequel, k denotes an integer, $k \geqq 2$.

LEMMA 5.2. *In the k-dimensional Euclidean space R^k, be given a rectangular coordinate system. Let D be any finite subset of R^k and let $D_1, \cdots, D_k$ denote the projections of D on the $(k-1)$-dimensional coordinate planes. Then*

$$[D] \leqq \sqrt[k-1]{[D_1][D_2] \cdots [D_k]}.$$

Here, the equality sign holds if and only if D coincides with the direct product of its projections on the k coordinate axes.

Proof of Theorem 5.1. Let the elements of A_i be enumerated in some arbitrary but fixed fashion

$$A_i = \{a_i(1), a_i(2), \cdots, a_i([A_i])\}$$

$(i = 1, \cdots, k)$. Consider an element $c \in E$. It admits one or more representations of the form

$$c = a_1(j_1) + a_2(j_2) + \cdots + a_k(j_k). \tag{5.3}$$

Choose j_1 as small as possible. If $j_1, \cdots, j_{i-1}$ have been chosen already, choose j_i as small as possible, $i = 1, \cdots, k$ (if the cancellation law does not hold, j_k is not yet determined by the other j_i). The resulting unique representation (5.3) of c will be called the canonical representation of c (cf. [1, p. 51]).

When (5.3) is the canonical representation of $c \in E$, put

$$\hat{c} = (j_1, \cdots, j_k).$$

In this way there corresponds to E a set D of lattice points $\hat{c}$ in R^k, $[D] = [E]$. By Lemma 5.2, it suffices to prove that

$$[D_i] \leqq [B_i^*] \qquad (i = 1, \cdots, k), \tag{5.3}$$

where $D_1, \cdots, D_k$ denote the projections of D on the $(k-1)$-dimensional coordinate planes. That is, D_i is the set of all points

$$d_i = (j_1, j_2, \cdots, j_{i-1}, 0, j_{i+1}, \cdots, j_k) \tag{5.4}$$

with integer coordinates, $1 \leqq j_s \leqq [A_s]$ if $s \neq i$, such that, for at least one choice of j_i, (5.3), is the canonical representation of some element $c \in E$.

In proving (5.3), it suffices to construct a $1:1$ mapping of D_i into B_i^* (i fixed). Namely, if $d_i \in D_i$ is given by (5.4), let

$$\phi(d_i) = a_1(j_1) + \cdots + a_{i-1}(j_{i-1}) + a_{i+1}(j_{i+1}) + \cdots + a_k(j_k).$$

Clearly, by (5.1) and the above property of d_i, one has $\phi(d_i) \in B_i^*$. Let

$$d_i' = (j_1', \cdots, j_{i-1}', 0, j_{i+1}', \cdots, j_k')$$

be a further point in D_i, such that $\phi(d_i) = \phi(d_i')$. We must prove that $d_i = d_i'$. Let $1 \leqq s \leqq k$, $s \neq i$, be such that $j_\lambda' = j_\lambda$ for $\lambda < s$, $\lambda \neq i$.

By $d_i \in D_i$, there exists an integer j_i such that (5.3) is the canonical representation of an element $c \in E$. By $\phi(d_i) = \phi(d_i')$,

$$c = a_i(j_i) + \phi(d_i') = a_1(j_1') + \cdots + a_k(j_k'),$$

where $j_i' = j_i$; thus, $j_\lambda' = j_\lambda$ for $\lambda < s$. It follows from the minimal character of j_s that $j_s' \geqq j_s$. Similarly, $j_s \geqq j_s'$; hence, $j_s' = j_s$. It follows that $j_\lambda' = j_\lambda$ for all $\lambda \neq i$; thus, $d_i' = d_i$.

If (5.2) holds with the equality sign then, by the last assertion of Lemma 5.2, D is the direct product of its projections in the coordinate axes; moreover, $[D_i] = [B_i^*]$ $(i = 1, \cdots, k)$. This easily implies the last assertion of Theorem 5.1.

THEOREM 5.3. *Let D be a given Lebesgue measurable subset of R^k. Let $D_1, \cdots, D_k$ denote the essential projections of D on the $(k-1)$-dimensional coordinate planes of a given rectangular coordinate system. Then*

$$\mu_k(D) \leqq \left[\prod_{i=1}^{k} \mu_{k-1}(D_i)\right]^{1/(k-1)}. \tag{5.5}$$

Here, the equality sign holds if and only if D differs by at most a set of μ_k-measure 0 from a direct product $A_1 \times \cdots \times A_k$($A_i$ denoting a measurable set on the ith coordinate axis).

Here, μ_r denotes r-dimensional Lebesgue measure. By the essential projection D_i of D we mean the set of points $d_i = (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_k)$ such that the line through d_i parallel to the ith coordinate axis intersects D in a set of positive linear outer measure. By Fubini, D_i is a Lebesgue measurable set. If

$$\begin{aligned} f(x) = f(x_1, \cdots, x_n) &= 1 \quad \text{if } x \in D, \\ &= 0 \quad \text{if } x \notin D, \end{aligned} \tag{5.6}$$

denotes the characteristic function of D then the (measurable) function

$$g_i(x) = g_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) = \operatorname*{ess\,sup}_{x_i} f(x_1, \cdots, x_k) \tag{5.7}$$

is precisely the characteristic function of D_i, $i = 1, \cdots, k$; (deleting from D a suitable set of measure zero, one may replace in (5.7) the essential supremum by the ordinary supremum).

Lemma 5.2 follows from Theorem 5.3 by describing about each point, of the finite set D in R^k, a small cube of fixed size having its edges parallel to the coordinate axes.

As far as I am aware, the isoperimetric inequality (5.5) is new. A related result due to Ohmann [2] gives as the upper bound on $\mu_k(D)^{1-1/k}$ the integral over all directions w of the $(k-1)$-dimensional Lebesgue measure of the w-projection of D, multiplied by a constant such the resulting upper bound is attained for a solid sphere.

Proof of Theorem 5.3. We may assume that $\mu_k(D) > 0$. The proof goes by induction with respect to k. We may assume that $k \geqq 3$, (the assertion being obvious for $k = 2$), and further that the assertion is true when k is replaced by $k - 1$. Finally (applying a suitable transformation $x_i' = c_i x_i$, $i = 1, \cdots, k$, to D), one may assume that

$$\int g_i(x)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_k = \mu_{k-1}(D_i) = 1 \tag{5.8}$$

($i = 1, \cdots, k$, in particular for $i = k$). Thus, in view of (5.7),

$$\int f(x_1, \cdots, x_k)\, dx_1\, dx_2 \cdots dx_{k-1} \leqq 1, \tag{5.9}$$

for almost all x_k. Moreover, by (5.7) and induction,

$$\int f(x_1, \cdots, x_k)\, dx_1 \cdots dx_{k-1} \leqq \left[\prod_{i=1}^{k-1} h_i(x_k)\right]^{1/(k-2)}, \tag{5.10}$$

for almost all x_k, where

$$h_i(x_k) = \int g_i(x)\,dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_{k-1} \qquad (i = 1, \cdots, k-1).$$

Note that, by (5.8),

$$\text{(5.11)} \qquad \int h_i(x_k)dx_k = 1 \qquad (i = 1, \cdots, k-1).$$

As is well known, $u_i \geqq 0$ $(i = 1, \cdots, k-1)$ implies

$$(u_1 u_2 \cdots u_{k-1})^{1/(k-1)} < (u_1 + \cdots + u_{k-1})/(k-1) = c,$$

say, unless the u_i are all equal to each other. Distinguishing between $c > 1$ and $c \leqq 1$, it follows that

$$\min(1, (u_1 \cdots u_{k-1})^{1/(k-2)}) \leqq (u_1 + \cdots + u_{k-1})/(k-1),$$

the equality sign holding if and only if either $u_i = 0$ for all i or $u_i = 1$ for all i. Replacing u_i by $h_i(x_k)$, $i = 1, \cdots, k-1$, it follows by (5.9) and (5.10) that

$$\text{(5.12)} \qquad \int f(x_1, \cdots, x_k)dx_1 \cdots dx_{k-1} \leqq (h_1(x_k) + \cdots + h_{k-1}(x_k))/(k-1),$$

for almost all x_k. Integrating with respect to x_k and using (5.11), one obtains that $\mu_k(D) \leqq 1$.

Suppose that $\mu_k(D) = 1$. Then for almost all x_k we have the following. First, either $h_i(x_k) = 0$ for all i (thus, the left-hand side of (5.12) is equal to zero), or $h_i(x_k) = 1$ for all i and the left-hand side of (5.12) is equal to 1, thus, by (5.7) and (5.8),

$$f(x_1, \cdots, x_{k-1}, x_k) = g_k(x_1, \cdots, x_{k-1}),$$

for almost all $(x_1, \cdots, x_{k-1})$. Moreover (in the latter case), (5.10) holds with the equality sign; hence, by induction,

$$g_k(x_1, \cdots, x_{k-1}) = \chi_{A_1}(x_1) \cdots \chi_{A_{k-1}}(x_{k-1}),$$

for almost all $(x_1, \cdots, x_{k-1})$. Here, A_i denotes a linear set with characteristic function $\chi_{A_i}(x)$, $i = 1, \cdots, k-1$. Letting A_k denote the set of those numbers x_k for which the left-hand side of (5.12) is equal to 1, it follows that

$$f(x) = f(x_1, \cdots, x_k) = \chi_{A_1}(x_1) \cdots \chi_{A_{k-1}}(x_{k-1})\chi_{A_k}(x_k),$$

for almost all $x = (x_1, \cdots, x_k)$.

By a different, somewhat less intuitive proof, the inequality (5.5) can be generalized as follows. For $i = 1, \cdots, k$, let

$$(\Omega_i, \mathscr{B}_i, \nu_i)$$

be a σ-finite measure space (ν_i a non-negative measure on the σ-field $\mathscr{B}_i$; we shall assume that $\mathscr{B}_i$ is complete relative to ν_i). Let their direct product be denoted as

$$(\Omega,\ \mathscr{B},\ \nu_1 \times \cdots \times \nu_k = \nu),$$

and let $\mathscr{B}'$ denote the completion of $\mathscr{B}$ relative to ν.

In the following, x_i runs through Ω_i; further, dx_i denotes the integration $\nu_i(dx_i)$ relative to x_i and ν_i, $i = 1, \cdots, k$.

Be given k *non-negative* functions

$$f_i(x) = f_i(x_1, \cdots, x_k) \geqq 0, \qquad i = 1, \cdots, k, \tag{5.13}$$

on Ω, each measurable relative to $\mathscr{B}'$. Define further

$$M = \int \prod_{i=1}^{k} f_i(x) dx_1 \cdots dx_k \tag{5.14}$$

and

$$M_i = \int \left[\operatorname*{ess\,sup}_{x_i} \int f_i(x)^{\ k-1} dx_1 \cdots dx_{i-1} \right] dx_{i+1} \cdots dx_k, \tag{5.15}$$

and

$$N = M^{1/k},\quad N_i = M_i^{1/(k-1)}, \qquad i = 1, \cdots, k. \tag{5.16}$$

THEOREM 5.4. *One has*

$$M \leqq (M_1 M_2 \cdots M_k)^{\ 1/(k-1)}; \tag{5.17}$$

equivalently,

$$N \leqq (N_1 N_2 \cdots N_k)^{1/k}. \tag{5.18}$$

The discussion of the equality sign is more complicated in the present case and will be omitted. Taking each function f_i equal to the characteristic function (5.6), Theorem 5.4 yields an inequality which is stronger than (5.5).

In proving Theorem 5.4, let us first consider the case $k = 2$. One has

$$\begin{aligned}\int f_1(x_1,x_2) f_2(x_1,x_2) dx_1 &\leqq [\operatorname*{ess\,sup}_{x_1} f_1(x_1,x_2)] \int f_2(x_1,x_2) dx_1 \\ &\leqq [\operatorname*{ess\,sup}_{x_1} f_1(x_1,x_2)] \operatorname*{ess\,sup}_{x_2} \int f_2(x_1,x_2) dx_1,\end{aligned}$$

the latter inequality holding at least for almost $[\nu_2]$ values x_2. Integrating with respect to x_2 one obtains (5.17).

Next, let $k \geqq 3$ be given, and suppose that Theorem 5.4 holds when k is replaced by $k - 1$. Put

$$I(x_k) = \int \prod_{i=1}^{k} f_i(x) dx_1 \cdots dx_{k-1}.$$

Letting

$$r = (k-1)/(k-2),\ \ s = k - 1,\ \ \text{thus,}\ \ \frac{1}{r} + \frac{1}{s} = 1,$$

it follows by Hölder's inequality that

$$I(x_k) \leqq \left[\int \left\{ \prod_{i=1}^{k-1} f_i(x) \right\}^r dx_1 \cdots dx_{k-1} \right]^{1/r} M_k^{1/(k-1)},$$

for almost $[\nu_k]$ all x_k; here, we also used (5.15) with $i = k$.

Applying (5.17) with k replaced by $k-1$ to the first integral and noting that $(k-2)r = k-1$, one obtains that

$$I(x_k) \leqq N_k \prod_{i=1}^{k-1} h_i(x_k)^{1/(k-1)} \tag{5.19}$$

for almost $[\nu_k]$ all x_k, where

$$h_i(x_k) = \int \left[\operatorname*{ess\,sup}_{x_i} \int f_i(x)^{k-1} dx_1 \cdots dx_{i-1} \right] dx_{i+1} \cdots dx_{k-1}$$

$(i = 1, \cdots, k-1)$. By (5.15) and (5.16),

$$\left\{ \int h_i(x_k)\, dx_k \right\}^{1/(k-1)} = M_i^{1/(k-1)} = N_i \qquad (i = 1, \cdots, k-1).$$

Hence, by (5.14), (5.19) and Hölder's inequality,

$$N^k = M = \int I(x_k) dx_k \leqq N_k (N_1 N_2 \cdots N_{k-1}).$$

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